

NAKAYAMA CATEGORIES AND GROUPOID QUANTIZATION

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ABSTRACT. We provide a precise description, albeit in the situation of standard categories, of the quantization functor Sum proposed by D.S. Freed, M.J. Hopkins, J. Lurie, and C. Teleman in a way enough abstract and flexible to suggest that an extension of the construction to the general context of (∞, n) -categories should indeed be possible. Our method is in fact based primarily on dualizability and adjunction data, and is well suited for the homotopical setting. The construction also sheds light on the need of certain rescaling automorphisms, and in particular on the nature and properties of the Nakayama isomorphism.

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1. INTRODUCTION

A k -extended n -dimensional \mathcal{C} -valued topological quantum field theory (TQFT for short) is a symmetric monoidal functor $Z: \text{Bord}_n^{n-k} \rightarrow \mathcal{C}$, from the (∞, k) -category of n -cobordisms extended down to codimension k [CS, Lu1] to a symmetric monoidal (∞, k) -category \mathcal{C} . For $k = 1$ one recovers the classical TQFT's as introduced in [At], while $k = n$ gives rise to *fully*

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extended TQFT's which, by the Cobordism Hypothesis [BD1, Lu1], we know being in correspondence with the fully dualizable objects of \mathcal{C} .

In [FHLT] the authors suggested the existence of an extended TQFT of Dijkgraaf-Witten type, associated to any given representation $V: X \rightarrow \mathcal{C}$ of a finite ∞ -groupoid X with values in a symmetric monoidal (∞, k) -category \mathcal{C} ; under the identification of ∞ -groupoids with homotopy types, V is equivalently seen as the datum of a local system on a topological space X of finite homotopy type, with values in \mathcal{C} . Such a theory appears as the composition of two monoidal functors

$$\begin{aligned} Bun_{k,V}: \text{Bord}_n^{n-k} &\rightarrow \text{Fam}_k(\mathcal{C}) \\ Sum_k: \text{Fam}_k(\mathcal{C}) &\rightarrow \mathcal{C} \end{aligned}$$

where $\text{Fam}_k(\mathcal{C})$ is an (∞, k) -category of representations, with objects spaces equipped with a functor to \mathcal{C} and morphisms the correspondences between these (see [Ha2, Lu1]). In the fully extended case the first functor associates to the point the given representation V , while Sum is the proper quantization functor carrying a representation $W: Y \rightarrow \mathcal{C}$ to its colimit. This can be thought of as taking the global sections of the local system W , or, from a more algebraic point of view, as the space of coinvariants for the action of the ∞ -groupoid Y on W .

Unfortunately in [FHLT] not many details are provided on how the above machinery should behave; the article in fact only addresses the above colimit construction and demands that colimits and limits in \mathcal{C} coincide in order to get a well defined functor. A better picture is given in the case when \mathcal{C} is the (∞, n) -category Alg_{n-1} (in the sense of [FHLT, Def. 7.1]) of iterated algebras and modules over Vect , and $V: X \rightarrow \text{Alg}_{n-1}$ is the trivial representation collapsing everything on the unit. In this situation, a detailed definition of the functor Sum has been given for $n = 2$ in [Mo1], while in [Mo2] also spaces endowed with a 2-cocycle have been considered; again, the author requires the existence of an isomorphism between right and left adjoints, given explicitly by the Nakayama isomorphism (cfr. [Ben]). It is worth noting that, despite the Cobordism Hypothesis, neither [FHLT] or [Mo1, Mo2] seem to make use of dualizability in their construction.

The primary goal of this paper is to provide a detailed construction of the functor Sum_1 and therefore of n -dimensional theories in the sense of Atiyah. The method we propose differs from the ones above in two main points:

- It does not assume the existence of any a priori given isomorphism between right and left adjoints.
- It relies almost only on existence of duals, in line with the Cobordism Hypothesis.

The main obstacles to the definition of the functor Sum appear when studying its behaviour at the level of morphisms. In fact, to a morphism α in $\text{Fam}_n(\mathcal{C})$ we could associate a morphism $Sum(\alpha)$ in \mathcal{C} via a pull-push procedure, as soon as we have a natural transformation (which is *not* required to be invertible) $f_* \rightarrow f_!$ from the right to the left adjoint of the pullback functor f^* , for any map $f: X \rightarrow Y$ of spaces.

A candidate γ_f is presented in Definition 3.6; the map was originally defined in [Hal] as long as certain projection formulas were satisfied, which actually always happens (Propositions 3.1 and 3.2) in presence of duals. Despite being very well behaved with respect to homotopy squares and external tensor products (as shown in sections 3.2 and 3.3), the map γ still lacks an essential feature. In fact, in order to make Sum into a functor, we need the assignment $f \mapsto \gamma_f$ to be functorial as well, in the sense that for any two composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ the following triangle

$$\begin{array}{ccc} & g!f_* & \\ \gamma_g \nearrow & & \searrow g!\gamma_f \\ g_*f_* & \xrightarrow{\gamma_{gf}} & g!f! \end{array}$$

should commute.

It can be seen that such condition is generally not verified, due to the possible non-triviality of the fibers of the maps f, g involved. As shown in Examples 4.4 and 4.5, when the target category \mathcal{C} is that of finite vector spaces over a positive characteristic field, one of the two paths might be the zero map while the other one is nonzero. Instead, in characteristic zero, the two ways of going from the right to the left adjoints differ by an invertible scalar.

One might then hope the assignment $f \mapsto \gamma_f$ to define a “projective functor”, and the discrepancy to result in a “cocycle condition”. This is actually the situation in the case of characteristic zero fields: we can in fact rescale the above map γ by a factor δ_f^{-1} related to the Baez-Dolan groupoid cardinality [BD2], so to neutralize the anomaly. In categorical terms, δ_f can be described via γ itself (Definition 4.7) and we can define a new map (the *Nakayama map*, Definition 4.8)

$$\nu_f: f_* \rightarrow f!$$

by multiplying γ_f by δ_f^{-1} . The best one can now achieve is to ask this new map to be functorial in the above sense. We will therefore define a *Nakayama category* (Definition 4.13) as a rigid category, in which for any map f of spaces the endomorphism δ_f is invertible and the assignment $f \mapsto \nu_f$ is functorial.

Under these hypotheses in Theorems 5.1 and 5.2 we will construct, by means of Kan extensions, two monoidal quantization functors

$$Sum: \text{Fam}_1(\mathcal{C}) \rightarrow \mathcal{C}$$

$$Prod: \text{Fam}_1(\mathcal{C}) \rightarrow \mathcal{C}$$

The map ν will then define a monoidal transformation between them and turn out to be an isomorphism (Proposition 5.3 and Corollary 5.5), thus answering an ambidexterity problem [HeL, HoL] in the context of categories with duals.

As one would expect, when \mathcal{C} is the category of finite dimensional vector spaces over a characteristic zero field, the map ν_f recovers the known Nakayama isomorphism, and justifies the choices made in [FHLT, Mo1, Mo2] as instances of a more general and abstract framework. In particular, it explains from a new point of view the obstruction to quantizing over fields of

positive characteristic.

Although the present paper only deals with standard categories and groupoids, the methods and ideas we propose are intrinsically homotopical, and formulated mainly in terms of adjunction and duality data via calculus of mates and Beck-Chevalley conditions. This suggests it should be possible to export them to the framework of higher categories so to build the proper n -dimensional theories which are the true aim of [FHLT], a topic which will be investigated elsewhere.

1.1. Conventions. While most of the constructions can be stated in the general case of non-symmetric monoidal categories, we will be interested into symmetric ones. By *monoidal category* we will therefore mean *symmetric monoidal category*. By space we will always mean ∞ -groupoid; in particular, also groupoids (i.e. homotopy 1-types) will be often referred to as spaces.

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2. PRELIMINARIES

We quickly review here below some notions and results, which are essential to the development of our machinery.

2.1. Duals. Recall first of all the notion of dualizable object in a monoidal category \mathcal{C} , which generalises and axiomatises the familiar concept of dual of a vector space. Dualizability is a key tool in this work, and more generally the main and possibly only ingredient to the construction of an extended Topological Quantum Field Theory, which is the ultimate goal of this paper.

Definition 2.1. *Let \mathcal{C} be a monoidal category and A an object of \mathcal{C} . An object A^d of \mathcal{C} is a dual of A if there are coevaluation and evaluation maps*

$$\begin{aligned} co : 1 &\rightarrow A^d \otimes A \\ ev : A \otimes A^d &\rightarrow 1 \end{aligned}$$

such that the compositions

$$\begin{aligned} A &\cong A \otimes 1 \xrightarrow{id \otimes co} A \otimes A^d \otimes A \xrightarrow{ev \otimes id} 1 \otimes A \cong A \\ A^d &\cong 1 \otimes A^d \xrightarrow{co \otimes id} A^d \otimes A \otimes A^d \xrightarrow{id \otimes ev} 1 \otimes A^d \cong A^d \end{aligned}$$

both equal the identity.

We say that an object A is dualizable if it has a dual. We say that a monoidal category \mathcal{C} has duals (or is a category with duals), if every object of \mathcal{C} is dualizable.

Notice that, being \mathcal{C} symmetric monoidal, A defines a dual to A^d .

As for the case of *Vect*-valued group representations, not only the monoidal structure is transferred to presheaf categories but also dualizability:

Proposition 2.2. *Let \mathcal{C} be a monoidal category with duals. Then, for any groupoid X , the functor category \mathcal{C}^X is a monoidal category with duals.*

The notion of dual can be nicely reformulated in terms of adjoints, in the following sense

Proposition 2.3. *Let \mathcal{C} be a monoidal category, and A an object of \mathcal{C} . If A has a dual A^d , then the coevaluation and evaluation maps define adjunctions*

$$\begin{aligned} A \otimes - &\dashv A^d \otimes - \\ - \otimes A^d &\dashv - \otimes A \end{aligned}$$

By universal property of the adjoints, it follows that any two duals to the same object A are isomorphic; in particular, one has $(A^d)^d \cong A$.

The latter is a very convenient way to deal with duals; by treating them as adjoint functors, in fact, we will be able to mix adjunction and dualizability data and place them in the framework of Beck-Chevalley transformations.

As a toy version of the cobordism hypothesis, finally, one can easily prove the following two results:

Proposition 2.4. *Let \mathcal{C}, \mathcal{D} be monoidal categories and $A \in \mathcal{C}$ an object having a dual A^d . Then*

- i) *For any monoidal functor $F: \mathcal{C} \rightarrow \mathcal{D}$, FA^d is a dual to FA .*
- ii) *If $\phi: F \rightarrow G$ is a monoidal transformation between monoidal functors $F, G: \mathcal{C} \rightarrow \mathcal{D}$, then $\phi_A: FA \rightarrow GA$ is an isomorphism.*

Corollary 2.5. *Let \mathcal{C}, \mathcal{D} be monoidal categories, and assume \mathcal{C} has duals. Then, any monoidal transformation $\phi: F \rightarrow G$ of monoidal functors from \mathcal{C} to \mathcal{D} is an isomorphism.*

2.2. Mates and Beck-Chevalley condition. We recollect now some results on the formation and properties of Beck-Chevalley transformed two-cells. This is a very helpful formalism, suited for the homotopical setting as well (see [Lu2, 4.7.5.13 et seqq.]), to handle adjunction data and functors, and to deal with Kan extensions.

Definition 2.6. *Let*

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ G \downarrow & \psi \nearrow & \downarrow G' \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \end{array}$$

be a lax square of functors. Assume that G and G' have left adjoints F, F' . The left mate of ψ is the map ψ_L defined as

$$F'H \xrightarrow{F'H\eta} F'HGF \xrightarrow{F'\psi} F'G'KF \xrightarrow{\epsilon'} KF$$

We say that the diagram satisfies the left Beck-Chevalley condition if the left mate of ψ is an isomorphism.

Definition 2.7. *Let*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \\ F \downarrow & \not\downarrow \phi & \downarrow F' \\ \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \end{array}$$

be an oplax square of functors. Assume that F and F' have right adjoints G, G' . The right mate of ϕ is the map ϕ_R defined as

$$HG \xrightarrow{\eta'} G'F'HG \xrightarrow{G'\phi} G'KFG \xrightarrow{G'K\varepsilon} G'K$$

We say that the diagram satisfies the right Beck-Chevalley condition if the right mate of ϕ is an isomorphism.

Particularly useful will be the nice behaviour of the above formalism with respect to both horizontal and vertical pasting of diagrams; the following lemma, and its dual version, will in fact be (ab)used all through this paper.

Lemma 2.8. i) *Let*

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{K} & \mathcal{D}' & \xrightarrow{K'} & \mathcal{D}'' \\ G \downarrow & \psi' \not\downarrow & G' \downarrow & \psi'' \not\downarrow & G'' \downarrow \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' & \xrightarrow{H'} & \mathcal{C}'' \end{array}$$

be lax squares, with G, G', G'' right adjoints. Let $\psi: H'HG \rightarrow G''K'K$ be the composition of ψ' and ψ'' . Then the left mate ψ_L of ψ is the composition of ψ'_L and ψ''_L .

ii) *Let*

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{L} & \mathcal{E}' \\ G_2 \downarrow & \psi'' \not\downarrow & \downarrow G'_2 \\ \mathcal{D} & \xrightarrow{K} & \mathcal{D}' \\ G_1 \downarrow & \psi' \not\downarrow & \downarrow G'_1 \\ \mathcal{C} & \xrightarrow{H} & \mathcal{C}' \end{array}$$

be lax squares, with G_1, G_2, G'_1, G'_2 right adjoints. Let $\psi: HG_1G_2 \rightarrow G'_1G'_2L$ be the composition of ψ'' and ψ' . Then ψ_L is the composition of ψ''_L and ψ'_L .

Clearly a specular statement holds for oplax squares and right mates; we leave to the reader the simple task to write it out.

It can be seen, as well, that the left and right Beck-Chevalley conditions interact in some cases and are essentially equivalent; more precisely we get

Proposition 2.9. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F_3} & \mathcal{B} \\ G_1 \downarrow & \theta \not\downarrow & \downarrow G_2 \\ \mathcal{C} & \xrightarrow{F_4} & \mathcal{D} \end{array}$$

be a lax square of functors, with $F_1 \dashv G_1, F_2 \dashv G_2, F_3 \dashv G_3, F_4 \dashv G_4$. Then the following are equivalent

- i) The diagram satisfies the left Beck-Chevalley condition
- ii) The diagram satisfies the right Beck-Chevalley condition

In the next pages we will be especially interested in diagrams arising from restriction functors. Recall that given $f: \mathcal{A} \rightarrow \mathcal{B}$ and a category \mathcal{C} , one has a restriction functor $f^*: \mathcal{C}^{\mathcal{B}} \rightarrow \mathcal{C}^{\mathcal{A}}$. Its left and right adjoints, if they exist, are given by Kan extensions and will be denoted respectively by $f_!$ and f_* ; a sufficient condition to their existence is given by cocompleteness, resp. completeness, of \mathcal{C} (see [ML]).

Definition 2.10. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{u} & \mathcal{A}' \\ p \downarrow & \searrow \phi & \downarrow q \\ \mathcal{B} & \xrightarrow{v} & \mathcal{B}' \end{array}$$

be a (op)lax square of functors. We say that the square is left (resp. right) exact if the corresponding diagram

$$\begin{array}{ccc} \mathcal{C}^{\mathcal{B}'} & \xrightarrow{v^*} & \mathcal{C}^{\mathcal{B}} \\ q^* \downarrow & \searrow \phi^* & \downarrow p^* \\ \mathcal{C}^{\mathcal{A}'} & \xrightarrow{u^*} & \mathcal{C}^{\mathcal{A}} \end{array}$$

satisfies the left (right) Beck-Chevalley condition, for any cocomplete (resp. complete) category \mathcal{C} . We say that it is an exact square if it is both left and right exact.

A natural question is whether one can find conditions for a square as above to be exact; a positive answer is given by pullbacks along Grothendieck fibrations, which specializes to the following, pivotal

Proposition 2.11. *Let*

$$\begin{array}{ccc} P & \longrightarrow & H \\ \downarrow & \searrow \phi & \downarrow f \\ K & \xrightarrow{g} & G \end{array}$$

be a homotopy pullback of groupoids. Then both the diagrams relative to ϕ and ϕ^{-1} are exact.

3. A COMPARISON MAP BETWEEN ADJOINTS

The present section focuses on the key ingredient of this paper, that is a natural morphism relating right and left Kan extensions along maps of groupoids. This is a map γ , first appeared in [Hal], which can be defined for any monoidal functor f^* having both left and right adjoints and satisfying certain projection formulas. We'll see that dualizability is a sufficient condition to the existence of γ , and via the calculus of mates we'll deduce several interesting properties exhibiting γ as well behaved w.r.t. homotopy exact squares and external tensor products. The results and constructions

given here will provide the background for the map properly used in our quantization functors, which we'll define and study in section 4.

The situation we are describing can be placed in the framework of Grothendieck's six operations (see for example [CD]) and, specifically, be understood as a Wirthmüller context. In view of the latter a map (Definition 3.10) similar to γ was previously defined in [FHM], and in fact the two can be shown to coincide (Proposition 3.11).

3.1. Projection formulas. The aforementioned map γ will arise thanks to certain ubiquitous projection morphisms λ and ρ . The key condition to existence of γ is invertibility of such maps, which we show here below being always satisfied thanks to dualizability. In this subsection we will moreover establish some useful lemmas concerning the maps λ and ρ , by means of Beck-Chevalley conditions and transformations.

Proposition 3.1 (Left Projection Formula). *Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor, having a left adjoint $f_!$. Then, there is a natural map*

$$\lambda: f_!(A \otimes f^* B) \rightarrow f_! A \otimes B$$

If \mathcal{C} has duals, then λ is an isomorphism.

Proof. Consider the square

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{- \otimes B} & \mathcal{C} \\ f^* \downarrow & \mu \nearrow & \downarrow f^* \\ \mathcal{D} & \xrightarrow{- \otimes f^* B} & \mathcal{D} \end{array}$$

and let λ be the left mate of μ . Invertibility of λ is just the left Beck-Chevalley condition for the above diagram. Since \mathcal{C} has duals, the horizontal functors have right adjoints by Propositions 2.3 and 2.4; therefore by Proposition 2.9 one can equivalently check the right Beck-Chevalley condition, which reduces to the isomorphism

$$f^*(A \otimes B^d) \xrightarrow{\mu^{-1}} f^* A \otimes f^* B^d$$

and we are done. \square

The dual notion and argument now gives

Proposition 3.2 (Right Projection Formula). *Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor, having a right adjoint f_* . Then, there is a natural map*

$$\rho: A \otimes f_* B \rightarrow f_*(f^* A \otimes B)$$

If \mathcal{C} has duals, then ρ is an isomorphism.

One could see that the projection morphisms satisfy several interesting properties such as compatibility with tensor product and composition, which can be understood as making the assignments $f \mapsto \lambda_f$, $f \mapsto \rho_f$ into “monoidal functors”. However we will omit them for brevity and only consider the following, stating an invariance of λ and ρ with respect to (op)lax squares, which will be needed later on. In particular, it implies stability under homotopy equivalences of functors, as one can easily check.

Proposition 3.3. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{g^*} & \mathcal{B} \\ h^* \downarrow & \phi \nearrow & \downarrow p^* \\ \mathcal{C} & \xrightarrow{q^*} & \mathcal{D} \end{array}$$

be a lax square of monoidal functors, commuting up to a monoidal transformation ϕ . Assume $h_! \dashv h^*, p_! \dashv p^*$. Then, the following diagram commutes

$$\begin{array}{ccccc} p_!(q^*A \otimes q^*h^*B) & \xrightarrow{\cong} & p_!q^*(A \otimes h^*B) & \xrightarrow{\phi_L} & g^*h_!(A \otimes h^*B) \\ p_!(id \otimes \phi) \downarrow & & & & \downarrow g^*\lambda \\ p_!(q^*A \otimes p^*g^*B) & \xrightarrow{\lambda} & p_!q^*A \otimes g^*B & \xrightarrow{\phi_L \otimes id} & g^*h_!A \otimes g^*B \xrightarrow{\cong} g^*(h_!A \otimes B) \end{array}$$

Proof. Consider the diagrams

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{- \otimes B} & \mathcal{A} & \xrightarrow{g^*} & \mathcal{B} \\ h^* \downarrow & \mu \nearrow & h^* \downarrow & \phi \nearrow & \downarrow p^* \\ \mathcal{C} & \xrightarrow{- \otimes h^*B} & \mathcal{C} & \xrightarrow{q^*} & \mathcal{D} \\ id \downarrow & \mu \nearrow & & & \downarrow id \\ \mathcal{C} & \xrightarrow{q^*} & \mathcal{D} & \xrightarrow{- \otimes q^*h^*B} & \mathcal{D} \end{array}$$

and

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{- \otimes B} & \mathcal{A} & \xrightarrow{g^*} & \mathcal{B} \\ id \downarrow & \mu \nearrow & id \downarrow & & \downarrow id \\ \mathcal{A} & \xrightarrow{g^*} & \mathcal{B} & \xrightarrow{- \otimes g^*B} & \mathcal{B} \\ h^* \downarrow & \phi \nearrow & p^* \downarrow & \mu \nearrow & \downarrow p^* \\ \mathcal{C} & \xrightarrow{q^*} & \mathcal{D} & \xrightarrow{- \otimes p^*g^*B} & \mathcal{D} \\ id \downarrow & id \otimes \phi \nearrow & id \downarrow & & \downarrow id \\ \mathcal{C} & \xrightarrow{q^* - \otimes q^*h^*B} & \mathcal{D} & & \mathcal{D} \end{array}$$

whose left mates give respectively the upper and lower paths, and finally apply Lemma 2.8. \square

By reversing arrows and using the pasting property for right mates we obtain

Proposition 3.4. *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{g^*} & \mathcal{B} \\ h^* \downarrow & \not\llcorner_{\pi} & \downarrow p^* \\ \mathcal{C} & \xrightarrow{q^*} & \mathcal{D} \end{array}$$

be an oplax square of monoidal functors, commuting up to a monoidal transformation π . Assume $h^* \dashv h_*$, $p^* \dashv p_*$. Then, the following diagram commutes

$$\begin{array}{ccccccc}
 g^*(A \otimes h_* B) & \xrightarrow{\cong} & g^* A \otimes g^* h_* B & \xrightarrow{id \otimes \pi_R} & g^* A \otimes p_* q^* B & \xrightarrow{\rho} & p_*(p^* g^* A \otimes q^* B) \\
 g^* \rho \downarrow & & & & & & \downarrow p_*(\pi \otimes id) \\
 g^* h_*(h^* A \otimes B) & \xrightarrow{\pi_R} & p_* q^*(h^* A \otimes B) & \xrightarrow{\cong} & p_*(q^* h^* A \otimes q^* B) & &
 \end{array}$$

3.2. Pre-Nakayama map. With the above results at hand, we can now introduce the Pre-Nakayama map γ ; once defined, we will study its behaviour with respect to homotopy commutative squares.

Definition 3.5. Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor between monoidal categories with duals, having both a right and a left adjoint. Define a natural map

$$\hat{\gamma}_f: f_*(A \otimes B) \rightarrow f_!(A \otimes B)$$

as

$$f_*(A \otimes B) \xrightarrow{f_*(\eta_! \otimes id)} f_*(f^* f_! A \otimes B) \xrightarrow{\lambda^{-1} \rho^{-1}} f_!(A \otimes f^* f_* B) \xrightarrow{f_!(id \otimes \varepsilon_*)} f_!(A \otimes B)$$

Definition 3.6 (Pre-Nakayama map). Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor between monoidal categories with duals, having both a right and a left adjoint. Define a natural transformation, the Pre-Nakayama map,

$$\gamma_f: f_* \rightarrow f_!$$

as

$$f_* A \cong f_*(A \otimes 1) \xrightarrow{\hat{\gamma}_f} f_!(A \otimes 1) \cong f_! A$$

According to expectations, the map $\hat{\gamma}$, and consequently γ , turns out to be well behaved w.r.t. homotopy commutative squares, by integrating Propositions 3.3 and 3.4 into

Lemma 3.7. Let

$$\begin{array}{ccc}
 & \mathcal{A} & \\
 g^* \swarrow & & \searrow h^* \\
 \mathcal{B} & \xRightarrow{\pi} & \mathcal{C} \\
 p^* \swarrow & & \searrow q^* \\
 & \mathcal{D} &
 \end{array}$$

be a diagram of monoidal functors between monoidal categories with duals, commuting up to a monoidal isomorphism π . Assume that

- the functors h^*, p^* have right and left adjoints
- the square satisfies the left Beck-Chevalley condition relative to π^{-1}
- the square satisfies the right Beck-Chevalley condition relative to π

Then, the following diagram commutes

$$\begin{array}{ccc}
 p_*(q^*A \otimes q^*B) & \xrightarrow{\hat{\gamma}_p} & p_!(q^*A \otimes q^*B) \\
 \cong \uparrow & & \downarrow \cong \\
 p_*q^*(A \otimes B) & & p_!q^*(A \otimes B) \\
 \pi_R \uparrow & & \downarrow (\pi^{-1})_L \\
 g^*h_*(A \otimes B) & \xrightarrow{g^*\hat{\gamma}_h} & g^*h_!(A \otimes B)
 \end{array}$$

Proof. The square arises by gluing

$$\begin{array}{ccccc}
 & & p_*(p^*p_!q^*A \otimes q^*B) & \xrightarrow{\rho^{-1}} & p_!q^*A \otimes p_*q^*B \\
 & \nearrow p_*(\eta_!^p \otimes id) & \downarrow p_*(p^*(\pi^{-1})_L \otimes id) & & \uparrow (\pi^{-1})_L^{-1} \otimes id \\
 p_*(q^*A \otimes q^*B) & & p_*(p^*g^*h_!A \otimes q^*B) & \xrightarrow{\rho^{-1}} & g^*h_!A \otimes p_*q^*B \\
 & \searrow p_*(q^*\eta_!^h \otimes id) & \uparrow p_*(\pi^{-1} \otimes id) & & \uparrow id \otimes \pi_R \\
 \cong \uparrow & & p_*(q^*h^*h_!A \otimes q^*B) & & \uparrow id \otimes \pi_R \\
 p_*q^*(A \otimes B) & & p_*q^*(h^*h_!A \otimes B) & & g^*h_!A \otimes g^*h_*B \\
 \pi_R \uparrow & & \pi_R \uparrow & & \cong \uparrow \\
 g^*h_*(A \otimes B) & \xrightarrow{g^*h_*(\eta_!^h \otimes id)} & g^*h_*(h^*h_!A \otimes B) & \xrightarrow{g^*\rho^{-1}} & g^*(h_!A \otimes h_*B)
 \end{array}$$

Proposition 3.4

with its specular diagram obtained via λ^{-1} . Propositions 3.3 and 3.4 and the Beck-Chevalley condition hypothesis on π and π^{-1} , together with a short computation involving the definition of mates and the axioms for monoidal structures, conclude the proof. \square

By applying the structure isomorphisms for monoidal categories and functors, one easily extends the above lemma to γ

Proposition 3.8. *Let*

$$\begin{array}{ccc}
 & \mathcal{A} & \\
 g^* \swarrow & & \searrow h^* \\
 \mathcal{B} & \xRightarrow{\pi} & \mathcal{C} \\
 p^* \swarrow & & \searrow q^* \\
 & \mathcal{D} &
 \end{array}$$

be a diagram of monoidal functors between monoidal categories with duals, commuting up to a monoidal isomorphism π . Assume that

- the functors h^*, p^* have right and left adjoints
- the square satisfies the left Beck-Chevalley condition relative to π^{-1}
- the square satisfies the right Beck-Chevalley condition relative to π

$$\begin{array}{ccc} p_* q^* A & \xrightarrow{\gamma_p} & p_! q^* A \\ \pi_R \uparrow & & \downarrow (\pi^{-1})_L \\ g^* h_* A & \xrightarrow{g^* \gamma_h} & g^* h_! A \end{array}$$

We shall now prove a technical result, which will be the keystone for the functoriality of the quantization functor Sum , and states the equivalence of the two possible “paths” one can walk along a homotopy commutative square.

$$\begin{array}{ccc} & \mathcal{A} & \\ g^* \swarrow & & \searrow h^* \\ \mathcal{B} & \xRightarrow{\pi} & \mathcal{C} \\ p^* \swarrow & & \searrow q^* \\ & \mathcal{D} & \end{array}$$

- the functors h^*, p^* have right and left adjoints
- the functors g^*, q^* have left adjoints
- the square satisfies the left Beck-Chevalley condition relative to π^{-1}
- the square satisfies the right Beck-Chevalley condition relative to π

$$\begin{array}{ccccccccccc}
g!g^* & \xrightarrow{g!\eta_*^p} & g!p_*p^*g^* & \xrightarrow{g!p_*\pi} & g!p_*q^*h^* & \xrightarrow{g!\gamma_p} & g!p!q^*h^* & \xrightarrow{\bar{\pi}} & h!q!q^*h^* & \xrightarrow{h!\varepsilon!^q} & h!h^* \\
& \searrow \eta_*^h & & & & & & & & & \\
& & h_*h^*g!g^* & \xrightarrow{\gamma_h} & h!h^*g!g^* & & & & & & \\
& & & & & \nearrow h!h^*\varepsilon!^g & & & & &
\end{array}$$
$$g_! p_! \xrightarrow{g_! p_! \eta_!^q} g_! p_! q^* q_! \xrightarrow{g_! (\pi^{-1})_L} g_! g^* h_! q_! \xrightarrow{\varepsilon_!^g} h_! q_!$$

As an aside observation, we compare a map given in [FHM] with the above γ . Again, we ask for a category with duals, even if the result holds (as for the previous ones) for functors satisfying the projection formulas.

Definition 3.10. Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor between categories with duals, having both a right and a left adjoint, and assume we have a map $\chi: f_*1 \rightarrow f_!1$. Define a natural transformation

$$\mu_\chi: f_* \rightarrow f_!$$

as

$$\begin{aligned} f_*X &\cong f_*X \otimes 1 \xrightarrow{id \otimes \eta_*} f_*X \otimes f_*f^*1 \xrightarrow{id \otimes \chi} f_*X \otimes f_!f^*1 \cong f_!f^*1 \otimes f_*X \\ &\xrightarrow{\lambda^{-1}} f_!(f^*1 \otimes f_*f_*X) \cong f_!f^*f_*X \xrightarrow{f_!\varepsilon_*} f_!X \end{aligned}$$

Proposition 3.11. Let $f^*: \mathcal{C} \rightarrow \mathcal{D}$ be a monoidal functor between categories with duals, having both a right and a left adjoint. Then

$$\mu_{\gamma_f(1)} = \gamma_f$$

Proof. The proposition follows from the properties of γ and of the projection maps. \square

3.3. Tensoring Kan extensions and pre-Nakayama maps. In the following we will focus on external tensor product of functors and how this is respected both by left and right Kan extensions, in the sense that we have isomorphisms

$$\begin{aligned} \mu_!: (f \times g)_!(A \boxtimes B) &\rightarrow f_!A \boxtimes g_!B \\ \mu_*: (f \times g)_*(A \boxtimes B) &\rightarrow f_*A \boxtimes g_*B \end{aligned}$$

relating the Kan extensions along a cartesian product of maps and the tensor product of the single Kan extensions. We will then see in Theorem 3.19 that the pre-Nakayama map agrees with such structure.

Let us first recall the external tensor product of functors, which might be understood as the categorical interpretation of external tensor product of group representations, and can be defined on presheaves valued in a monoidal category as follows

Definition 3.12. Let \mathcal{C} be a monoidal category, and $A \in \mathcal{C}^X, B \in \mathcal{C}^Y$ functors. The external tensor product of A and B is the functor $A \boxtimes B \in \mathcal{C}^{X \times Y}$

$$A \boxtimes B := p_X^*A \otimes p_Y^*B$$

where p_X, p_Y are the obvious projections.

One has natural “associativity” and “twist” isomorphisms

$$\begin{aligned} \bar{\alpha}: (A \boxtimes B) \boxtimes C &\rightarrow A \boxtimes (B \boxtimes C) \\ \bar{\tau}: A \boxtimes B &\rightarrow B \boxtimes A \end{aligned}$$

given respectively by

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{id \times p_Y^* \times p_Z^*} & \mathcal{C}^X \times \mathcal{C}^{Y \times Z} \times \mathcal{C}^{Y \times Z} & \xrightarrow{id \times \otimes} & \mathcal{C}^X \times \mathcal{C}^{Y \times Z} \\
\downarrow p_X^* \times p_Y^* \times id & & \downarrow p_X^* \times p_Y^* \times p_Z^* \times p_{Y \times Z}^* & \xRightarrow{\mu^{-1}} & \downarrow p_X^* \times p_{Y \times Z}^* \\
\mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} \times \mathcal{C}^Z & \xrightarrow{p_X^* \times p_Y^* \times p_Z^*} & \mathcal{C}^{X \times Y \times Z} \times \mathcal{C}^{X \times Y \times Z} \times \mathcal{C}^{X \times Y \times Z} & \xrightarrow{id \times \otimes} & \mathcal{C}^{X \times Y \times Z} \times \mathcal{C}^{X \times Y \times Z} \\
\downarrow \otimes \times id & \xRightarrow{\mu} & \downarrow \otimes \times id & \xRightarrow{\alpha} & \downarrow \otimes \\
\mathcal{C}^{X \times Y} \times \mathcal{C}^Z & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y \times Z} \times \mathcal{C}^{X \times Y \times Z} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y \times Z}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y} \\
\downarrow id & & \downarrow \cong & \nearrow \mu & \downarrow \\
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{Y \times X} \times \mathcal{C}^{Y \times X} & \xrightarrow{\otimes} & \mathcal{C}^{Y \times X} \\
\downarrow \cong & & \downarrow id & \nearrow \tau & \downarrow id \\
\mathcal{C}^Y \times \mathcal{C}^X & \xrightarrow{p_Y^* \times p_X^*} & \mathcal{C}^{Y \times X} \times \mathcal{C}^{Y \times X} & \xrightarrow{\otimes} & \mathcal{C}^{Y \times X}
\end{array}$$

where we considered the obvious twisting isomorphisms for the cartesian product.

From now \mathcal{C} will be a fixed monoidal category with duals, closed under colimits and limits indexed by groupoids. This will ensure existence of both left and right Kan extensions along any map f between groupoids, which we denote by $f_!$ and f_* ; groupoids will be denoted by capital letters M, N, \dots, X, Y, \dots . Also, recall from Proposition 2.2 that the functor categories \mathcal{C}^X are symmetric monoidal with duals.

The primary goal is to define isomorphisms $\mu_!, \mu_*$. One should think of the following as making the Kan extension constructions into symmetric monoidal functors.

Proposition 3.13. *Given maps $f: M \rightarrow X$ and $g: N \rightarrow Y$, for any $A \in \mathcal{C}^M, B \in \mathcal{C}^N$ we have natural isomorphisms*

$$\begin{aligned}
\mu_! &: (f \times g)_!(A \boxtimes B) \rightarrow f_! A \boxtimes g_! B \\
\mu_* &: (f \times g)_*(A \boxtimes B) \rightarrow f_* A \boxtimes g_* B
\end{aligned}$$

Proof. We prove the existence of the isomorphism $\mu_!$, being the other dual. Consider the left mate resulting from the following diagram

$$\begin{array}{ccccc}
 & & \boxtimes & & \\
 & \searrow & & \nearrow & \\
 \mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y} \\
 \downarrow f^* \times id & & \downarrow (f \times id)^* \times id & \nearrow \mu & \downarrow (f \times id)^* \\
 \mathcal{C}^M \times \mathcal{C}^Y & \xrightarrow{p_M^* \times p_Y^*} & \mathcal{C}^{M \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes (f \times id)^*} & \mathcal{C}^{M \times Y} \\
 \downarrow id & & \downarrow id & & \downarrow id \\
 \mathcal{C}^M \times \mathcal{C}^Y & \xrightarrow{p_M^* \times p_Y^*} & \mathcal{C}^{M \times Y} \times \mathcal{C}^{M \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{M \times Y} \\
 \downarrow id \times g^* & \nearrow id & \downarrow id \times (id \times g)^* & \nearrow \mu & \downarrow (id \times g)^* \\
 \mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{p_M^* \times p_N^*} & \mathcal{C}^{M \times Y} \times \mathcal{C}^{M \times N} & \xrightarrow{(id \times g)^* \otimes} & \mathcal{C}^{M \times N} \\
 & \searrow & & \nearrow & \\
 & & \boxtimes & &
 \end{array}$$

The left sided squares give rise to isos on each factor thanks to Proposition 2.11. The right sided squares reduce, pointwise, to the maps λ of Proposition 3.1. \square

A simple argument then proves associativity of $\mu_!$ and μ_*

Proposition 3.14. *Given maps $M \xrightarrow{f} X$, $N \xrightarrow{g} Y$, $O \xrightarrow{h} Z$, for any $A \in \mathcal{C}^M$, $B \in \mathcal{C}^N$, $C \in \mathcal{C}^O$ the following diagrams commute*

$$\begin{array}{ccccc}
 (f \times g \times h)_!(A \boxtimes B) \boxtimes C & \xrightarrow{\mu_!} & (f \times g)_!(A \boxtimes B) \boxtimes h_!C & \xrightarrow{\mu_! \boxtimes id} & (f_!A \boxtimes g_!B) \boxtimes h_!C \\
 \cong \downarrow & & & & \downarrow \cong \\
 (f \times g \times h)_!(A \boxtimes (B \boxtimes C)) & \xrightarrow{\mu_!} & f_!A \boxtimes (g \times h)_!(B \boxtimes C) & \xrightarrow{id \boxtimes \mu_!} & f_!A \boxtimes (g_!B \boxtimes h_!C)
 \end{array}$$

$$\begin{array}{ccccc}
 (f \times g \times h)_*(A \boxtimes (B \boxtimes C)) & \xrightarrow{\mu_*} & f_*A \boxtimes (g \times h)_*(B \boxtimes C) & \xrightarrow{id \boxtimes \mu_*} & f_*A \boxtimes (g_*B \boxtimes h_*C) \\
 \cong \downarrow & & & & \downarrow \cong \\
 (f \times g \times h)_*(A \boxtimes B) \boxtimes C & \xrightarrow{\mu_*} & (f \times g)_*(A \boxtimes B) \boxtimes h_*C & \xrightarrow{\mu_* \boxtimes id} & (f_*A \boxtimes g_*B) \boxtimes h_*C
 \end{array}$$

Proof. Being the two diagrams dual to each other, we give just the proof for the first. The upper and lower paths are, respectively, the left mates of

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{id \times \boxtimes} & \mathcal{C}^X \times \mathcal{C}^{Y \times Z} & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y \times Z} \\
\downarrow id & & \bar{\alpha} \Uparrow & & \downarrow id^* \\
\mathcal{C}^X \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{\boxtimes \times id} & \mathcal{C}^{X \times Y} \times \mathcal{C}^Z & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y \times Z} \\
\downarrow f^* \times id \times id & \Uparrow & \downarrow (f \times id)^* \times id & \Uparrow & \downarrow (f \times id \times id)^* \\
\mathcal{C}^M \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{\boxtimes \times id} & \mathcal{C}^{M \times Y} \times \mathcal{C}^Z & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times Y \times Z} \\
\downarrow id \times g^* \times id & \Uparrow & \downarrow (id \times g)^* \times id & \Uparrow & \downarrow (id \times g \times id)^* \\
\mathcal{C}^M \times \mathcal{C}^N \times \mathcal{C}^Z & \xrightarrow{\boxtimes \times id} & \mathcal{C}^{M \times N} \times \mathcal{C}^Z & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N \times Z} \\
\downarrow id \times id \times h^* & \Uparrow & \downarrow (id \times id) \times h^* & \Uparrow & \downarrow (id \times id \times h)^* \\
\mathcal{C}^M \times \mathcal{C}^N \times \mathcal{C}^O & \xrightarrow{\boxtimes \times id} & \mathcal{C}^{M \times N} \times \mathcal{C}^O & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N \times O}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{id \times \boxtimes} & \mathcal{C}^X \times \mathcal{C}^{Y \times Z} & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y \times Z} \\
\downarrow f^* \times id \times id & \Uparrow & \downarrow f^* \times (id \times id) & \Uparrow & \downarrow (f \times id \times id)^* \\
\mathcal{C}^M \times \mathcal{C}^Y \times \mathcal{C}^Z & \xrightarrow{id \times \boxtimes} & \mathcal{C}^M \times \mathcal{C}^{Y \times Z} & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times Y \times Z} \\
\downarrow id \times g^* \times id & \Uparrow & \downarrow id \times (g \times id)^* & \Uparrow & \downarrow (id \times g \times id)^* \\
\mathcal{C}^M \times \mathcal{C}^N \times \mathcal{C}^Z & \xrightarrow{id \times \boxtimes} & \mathcal{C}^M \times \mathcal{C}^{N \times Z} & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N \times Z} \\
\downarrow id \times id \times h^* & \Uparrow & \downarrow id \times (id \times h)^* & \Uparrow & \downarrow (id \times id \times h)^* \\
\mathcal{C}^M \times \mathcal{C}^N \times \mathcal{C}^O & \xrightarrow{id \times \boxtimes} & \mathcal{C}^M \times \mathcal{C}^{N \times O} & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N \times O} \\
\downarrow id & & \bar{\alpha} \Uparrow & & \downarrow id
\end{array}$$

where we pasted the diagrams appearing in Definition 3.12 and Proposition 3.13. They are easily seen to be equivalent by definition of \boxtimes and the transformations involved, and monoidality of the restriction functors. Lemma 2.8 then concludes the proof. \square

Again, via the pasting lemmas, it is not difficult to check the symmetry axiom.

Proposition 3.15. *Given maps $M \xrightarrow{f} X$, $N \xrightarrow{g} Y$, for any $A \in \mathcal{C}^M$, $B \in \mathcal{C}^N$ the following diagrams commute*

$$\begin{array}{ccc}
(f \times g)_!(A \boxtimes B) & \xrightarrow{\mu_!} & f_! A \boxtimes g_! B \\
\cong \downarrow & & \downarrow \cong \\
(g \times f)_!(B \boxtimes A) & \xrightarrow{\mu_!} & g_! B \boxtimes f_! A
\end{array}$$

$$\begin{array}{ccc}
(f \times g)_*(A \boxtimes B) & \xrightarrow{\mu_*} & f_*A \boxtimes g_*B \\
\cong \downarrow & & \downarrow \cong \\
(g \times f)_*(B \boxtimes A) & \xrightarrow{\mu_*} & g_*B \boxtimes f_*A
\end{array}$$

The same arguments also show that $\mu_!, \mu_*$ behave functorially

Proposition 3.16. *Let $P \xrightarrow{p} M \xrightarrow{f} X$ and $Q \xrightarrow{q} N \xrightarrow{g} Y$ be pairs of composable maps. Then for any $A \in \mathcal{C}^P, B \in \mathcal{C}^Q$, the diagrams*

$$\begin{array}{ccc}
(f \times g)_!(p \times q)_!(A \boxtimes B) & \xrightarrow{(f \times g)_! \mu_!} & (f \times g)_!(p_!A \boxtimes q_!B) \xrightarrow{\mu_!} f_!p_!A \boxtimes g_!q_!B \\
& \searrow \mu_! & \\
(f \times g)_*(p \times q)_*(A \boxtimes B) & \xrightarrow{(f \times g)_* \mu_*} & (f \times g)_*(p_*A \boxtimes q_*B) \xrightarrow{\mu_*} f_*p_*A \boxtimes g_*q_*B
\end{array}$$

commute.

Proof. The first composition results by pasting the left mates of the inner squares in the following diagram, while the lower map is the left mate of the outer diagram:

$$\begin{array}{ccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y} \\
f^* \times g^* \downarrow & \nearrow & \downarrow (f \times g)^* \\
\mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N} \\
p^* \times q^* \downarrow & \nearrow & \downarrow (p \times q)^* \\
\mathcal{C}^P \times \mathcal{C}^Q & \xrightarrow{\boxtimes} & \mathcal{C}^{P \times Q}
\end{array}$$

□

Finally, as one would expect, stability under homotopy equivalences is an immediate consequence of the Beck-Chevalley formalism.

Proposition 3.17. *Let*

$$\begin{array}{ccc}
M & \xrightarrow{f} & X \\
& \Downarrow \phi & \\
M & \xrightarrow{f'} & X
\end{array}
\quad
\begin{array}{ccc}
N & \xrightarrow{g} & X \\
& \Downarrow \psi & \\
N & \xrightarrow{g'} & X
\end{array}$$

be natural transformations of maps between groupoids. Then for any $A \in \mathcal{C}^M, B \in \mathcal{C}^N$, the following diagrams commute

$$\begin{array}{ccc}
(f' \times g')_!(A \boxtimes B) & \xrightarrow{\mu_!} & f'_! A \boxtimes g'_! B \\
(\phi \times \psi)_L \downarrow & & \downarrow \phi_L \boxtimes \psi_L \\
(f \times g)_!(A \boxtimes B) & \xrightarrow{\mu_!} & f_! A \boxtimes g_! B
\end{array}$$

$$\begin{array}{ccc}
(f' \times g')_*(A \boxtimes B) & \xrightarrow{\mu_*} & f'_* A \boxtimes g'_* B \\
(\phi \times \psi)_R \downarrow & & \downarrow \phi_R \boxtimes \psi_R \\
(f \times g)_*(A \boxtimes B) & \xrightarrow{\mu_*} & f_* A \boxtimes g_* B
\end{array}$$

Proof. The upper and lower paths in the first square are given by the left mates of the following diagrams

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{id} & \mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y} \\
f^* \times g^* \downarrow & \nearrow \phi^* \times \psi^* & f'^* \times g'^* \downarrow & \nearrow & \downarrow (f' \times g')^* \\
\mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{id} & \mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N}
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y} & \xrightarrow{id} & \mathcal{C}^{X \times Y} \\
f^* \times g^* \downarrow & \nearrow & (f \times g)^* \downarrow & \nearrow (\phi \times \psi)^* & \downarrow (f' \times g')^* \\
\mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N} & \xrightarrow{id} & \mathcal{C}^{M \times N}
\end{array}$$

which are easily seen to be equivalent, for ϕ and ψ induce monoidal transformations.

Commutativity of the second square is proved similarly. \square

In view of Theorems 5.1 and 5.2 we also provide the following

Proposition 3.18. *Let $f: M \rightarrow X$ and $g: N \rightarrow Y$ be maps. Then for any $A \in \mathcal{C}^X, B \in \mathcal{C}^Y$, the diagrams*

$$\begin{array}{c}
(f \times g)_!(f \times g)^*(A \boxtimes B) \xrightarrow{\cong} (f \times g)_!(f^* A \boxtimes g^* B) \xrightarrow{\mu_!} f_! f^* A \boxtimes g_! g^* B \xrightarrow{\varepsilon \boxtimes \varepsilon} A \boxtimes B \\
\searrow \varepsilon \qquad \qquad \qquad \nearrow
\end{array}$$

$$\begin{array}{c}
A \boxtimes B \xrightarrow{\eta} (f \times g)_*(f \times g)^*(A \boxtimes B) \xrightarrow{\cong} (f \times g)_*(f^* A \boxtimes g^* B) \xrightarrow{\mu_*} f_* f^* A \boxtimes g_* g^* B \\
\searrow \eta \boxtimes \eta \qquad \qquad \qquad \nearrow
\end{array}$$

commute.

Proof. A simple computation shows that the diagram

$$\begin{array}{ccccccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{id} & \mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y} \\
\downarrow id & & \downarrow f^* \times g^* & & \downarrow \mu & & \downarrow (f \times g)^* \\
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{f^* \times g^*} & \mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{p_M^* \times p_N^*} & \mathcal{C}^{M \times N} \times \mathcal{C}^{M \times N} & \xrightarrow{\otimes} & \mathcal{C}^{M \times N} \\
\downarrow id & & \downarrow id & & \downarrow id & & \downarrow id \\
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{(f \times g)^* \times (f \times g)^*} & \mathcal{C}^{M \times N} \times \mathcal{C}^{M \times N} & \xrightarrow{\otimes} & \mathcal{C}^{M \times N} \\
\downarrow id & & \downarrow id & & \downarrow \mu^{-1} & & \downarrow id \\
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y} & \xrightarrow{(f \times g)^*} & \mathcal{C}^{M \times N}
\end{array}$$

whose left mate gives the first composition, is equivalent to the diagram

$$\begin{array}{ccc}
\mathcal{C}^{X \times Y} & \xrightarrow{id} & \mathcal{C}^{X \times Y} \\
id \downarrow & & \downarrow (f \times g)^* \\
\mathcal{C}^{X \times Y} & \xrightarrow{(f \times g)^*} & \mathcal{C}^{M \times N}
\end{array}$$

giving the counit ε .

The same argument proves the other statement. \square

We can now get back to our pre-Nakayama map, and conclude the section with a compatibility theorem

Theorem 3.19. *Let $f: M \rightarrow X$ and $g: N \rightarrow Y$ be maps. Then for any $A \in \mathcal{C}^M, B \in \mathcal{C}^N$*

$$\begin{array}{ccc}
(f \times g)_*(A \boxtimes B) & \xrightarrow{\mu_*} & f_* A \boxtimes g_* B \\
\gamma_{f \times g} \downarrow & & \downarrow \gamma_f \boxtimes \gamma_g \\
(f \times g)_!(A \boxtimes B) & \xrightarrow{\mu_!} & f_! A \boxtimes g_! B
\end{array}$$

commutes.

Proof. By definition, $\mu_!$ and μ_* are the left and right mates for the diagram

$$\begin{array}{ccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{\boxtimes} & \mathcal{C}^{X \times Y} \\
f^* \times g^* \downarrow & \nearrow & \downarrow (f \times g)^* \\
\mathcal{C}^M \times \mathcal{C}^N & \xrightarrow{\boxtimes} & \mathcal{C}^{M \times N}
\end{array}$$

Since all functors and transformations involved are monoidal, and since the pre-Nakayama map relative to $f^* \times g^*$ is $\gamma_f \times \gamma_g$, we can apply Proposition 3.8. \square

4. NAKAYAMA CATEGORIES

As already mentioned in the introduction and shown below, the assignment $f \mapsto \gamma_f$ fails to be functorial in general. The latter property will be essential to the construction of the quantization functors, in order to basically reduce our problem to Theorem 3.9.

The next pages are therefore devoted to correct this issue, by building a new map ν , and to the definition of what we call a “Nakayama category”; this can be understood as an abstract version of finite vector spaces over a zero-characteristic field, and will encode the hypotheses needed to build the functor Sum .

We give first the following, which allows us to reduce the assumptions on our category \mathcal{C}

Proposition 4.1. *Let \mathcal{C} and \mathcal{D} be monoidal categories with duals, and $f^*: \mathcal{C} \rightarrow \mathcal{D}$ a monoidal functor. Then, f^* has a left adjoint if and only if it has a right adjoint.*

Proof. Assume that f^* has a left adjoint, then we have a chain of isomorphisms

$$\mathcal{D}(f^*B, A) \cong \mathcal{D}(A^d, (f^*B)^d) \cong \mathcal{D}(A^d, f^*B^d) \cong \mathcal{C}(f_!A^d, B^d) \cong \mathcal{C}(B, (f_!A^d)^d)$$

expressing f^* as a left adjoint to the functor

$$A \mapsto (f_!A^d)^d$$

The other direction of the equivalence is proved similarly. \square

From now on we will be interested in categories \mathcal{C}^X of functors from essentially finite groupoids, by which we mean

Definition 4.2. *A groupoid X is said essentially finite if the set $\pi_0 X$ of isomorphism classes of objects of X is finite, and for each pair of objects $x, y \in X$ the set $X(x, y)$ is finite.*

It can be seen that the Quillen equivalence between homotopy 1-types and groupoids (see for example [JT, Section 2.2]) restricts to a Quillen equivalence between the categories of finite homotopy 1-types and of essentially finite groupoids. We will denote both categories by \mathbf{Gpd}_{fin} .

Also we will be of course concerned with Kan extensions along groupoid maps, which we know exist whenever \mathcal{C} satisfies the following

Definition 4.3. *A category \mathcal{C} admits essentially finite (co)limits, if it admits (co)limits indexed by objects of \mathbf{Gpd}_{fin} .*

The simplest example of essentially finite groupoids is given by the classifying space BG of a finite group G , while finite vector spaces are clearly closed under essentially finite (co)limits and, having duals, allow us to define the map γ of the previous section. It is in this context that one can easily check the failure of γ to satisfy functoriality. One would in fact expect that

for composable maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ of groupoids the diagram

$$\begin{array}{ccccc} g_* f_* & \xrightarrow{g_* \gamma_f} & g_* f! & \xrightarrow{\gamma_g} & g! f! \\ \cong \downarrow & & & & \downarrow \cong \\ (gf)_* & \xrightarrow{\gamma_{gf}} & & & (gf)! \end{array}$$

commutes.

With reference to Theorems 6.1, 6.2 and 6.3, we have instead:

Example 4.4. Let C_n be the cyclic group of order n with generator g , and consider the composition

$$\{*\} \xrightarrow{s} BC_n \xrightarrow{t} \{*\}$$

Letting k the trivial representation of the point in $Vect_k$, one gets a diagram

$$\begin{array}{ccc} Hom_{C_n}(k, Hom_k(k[C_n], k)) & \xrightarrow{\gamma_t \circ t_* \gamma_s} & k \otimes_{C_n} k[C_n] \otimes k \\ \cong \downarrow & & \uparrow \cong \\ Hom_k(k, k) & \xrightarrow{\gamma_{ts}} & k \otimes k \end{array}$$

If $\phi \in Hom_{C_n}(k, Hom_k(k[C_n], k))$, so that $\psi = \phi(1)$ is constant on the elements of C_n with value $\psi(g^j) = \bar{\psi}$, then ϕ is taken by the horizontal morphism to

$$n \cdot 1 \otimes_{C_n} 1 \otimes \bar{\psi}$$

while via the lower path the factor n does not arise, and one has just

$$1 \otimes_{C_n} 1 \otimes \bar{\psi}$$

It is clear then, that such discrepancy can not be eliminated whenever the characteristic of k divides n . In characteristic zero, instead, one can still hope to rescale the maps to avoid the problem.

One should notice as well that the anomaly highlighted above varies locally at each connected component of the spaces involved, as explained in the following

Example 4.5. Let C_m, C_n be the cyclic groups of order m and n with generators h and g , and consider the composition

$$\{x, y\} \xrightarrow{s} BC_m \sqcup BC_n \xrightarrow{t} \{*\}$$

where s sends x, y respectively to the unique zero-cells of BC_m and BC_n . We now get two maps

$$\begin{array}{c} Hom_{C_m}(k, Hom_k(k[C_m], k)) \oplus Hom_{C_n}(k, Hom_k(k[C_n], k)) \\ \gamma_{ts} \left(\begin{array}{c} \downarrow \qquad \downarrow \\ (k \otimes_{C_m} k[C_m] \otimes k) \oplus (k \otimes_{C_n} k[C_n] \otimes k) \end{array} \right. \gamma_t \circ t_* \gamma_s \end{array}$$

Given a pair of morphisms (ϕ_m, ϕ_n) (sending 1 to functions $\psi_m(h^i) = \bar{\psi}_m$, $\psi_n(g^j) = \bar{\psi}_n$) we obtain via the rightmost map the value

$$(m \cdot 1 \otimes_{C_m} 1 \otimes \bar{\psi}_m, \quad n \cdot 1 \otimes_{C_n} 1 \otimes \bar{\psi}_n)$$

The other map, as before, ignores the contributions and gives

$$(1 \otimes_{C_m} 1 \otimes \bar{\psi}_m, \quad 1 \otimes_{C_n} 1 \otimes \bar{\psi}_n)$$

As for the previous example, in non-zero characteristic there is no way to cure the problem.

4.1. Weights and the Nakayama map. We now try make sense of the discrepancies appeared in Examples 4.4 and 4.5 and define a new map ν_f out of γ_f , with similar properties.

First, in view of Proposition 2.11, we can give the following definition of Kan extensions along maps of groupoids

Definition 4.6. *If $f: X \rightarrow Y$ is a map between essentially finite groupoids, let*

$$\begin{array}{ccc} P_y & \xrightarrow{p} & X \\ f_y \downarrow & \nearrow \pi & \downarrow f \\ \{*\} & \xrightarrow{y} & Y \end{array}$$

*be the homotopy fiber of f at the point y . Given $V: X \rightarrow \mathcal{C}$, the value at y of the left (resp. right) Kan extension $f_!V$ (f_*V) along f , is given by the colimit (resp. limit) of p^*V .*

We can define an endomorphism of the trivial representation 1 on P_y as follows

Definition 4.7. *Let $f: X \rightarrow Y$ be a map between essentially finite groupoids. On each connected component $\hat{P} \hookrightarrow P_y$ of the fiber*

$$\begin{array}{ccc} P_y & \xrightarrow{p} & X \\ f_y \downarrow & \nearrow \pi & \downarrow f \\ \{*\} & \xrightarrow{y} & Y \end{array}$$

at y , the (essentially) unique map $\{\} \xrightarrow{\iota} \hat{P}$ defines an endomorphism*

$$1 \xrightarrow{\eta_*^\iota} \iota_* \iota^* 1 \xrightarrow{\gamma_\iota} \iota_! \iota^* 1 \xrightarrow{\varepsilon_!^\iota} 1$$

of the trivial representation 1 on \hat{P} . The collection of such maps gives then an endomorphism of $1 \in \mathcal{C}^{P_y}$

$$\delta_f: 1 \rightarrow 1$$

*and therefore a natural endomorphism of p^**

$$p^*V \cong p^*V \otimes 1 \xrightarrow{id \otimes \delta_f} p^*V \otimes 1 \cong p^*V$$

which we shall denote again by δ_f .

Recall that for an object X in a symmetric monoidal category with duals \mathcal{C} , the dimension of X is the endomorphism

$$1_{\mathcal{C}} \xrightarrow{co} X \otimes X^d \xrightarrow{ev} 1_{\mathcal{C}}$$

One can easily prove that, at each connected component $\hat{P} \hookrightarrow P_y$, δ_f is the multiplication by the dimension of $\iota_1 1$. Placing us back in the situation of Example 4.5, the endomorphism δ_t associated to $t: BC_m \sqcup BC_n \rightarrow \{*\}$ multiplies the vectors relative to BC_j ($j = m, n$) exactly by j .

It makes sense, then, to give the following

Definition 4.8 (Nakayama map). *Let $f: X \rightarrow Y$ be a map of essentially finite groupoids, and assume that δ_f is invertible.*

Define a natural transformation, the Nakayama map,

$$\nu_f: f_* \rightarrow f_!$$

as

$$f_* V \xrightarrow{\gamma_f} f_! V \xrightarrow{f_! \delta_f^{-1}} f_! V$$

Notice that ν_f could as well be defined as

$$f_* V \xrightarrow{f_* \delta_f^{-1}} f_* V \xrightarrow{\gamma_f} f_! V$$

We can now show that most properties of γ_f , proved in the previous section, are satisfied by ν_f as well.

Proposition 4.9. *Let*

$$\begin{array}{ccc} & P & \\ p \swarrow & & \searrow q \\ M & \xRightarrow{\pi} & N \\ g \searrow & & \swarrow h \\ & Y & \end{array}$$

be a homotopy pullback of essentially finite groupoids. Let \mathcal{C} be a symmetric monoidal category with duals closed under essentially finite (co)limits, and assume that δ_p, δ_h are invertible; then, the following diagram commutes

$$\begin{array}{ccc} p_* q^* V & \xrightarrow{\nu_p} & p_! q^* V \\ \pi_R \uparrow & & \downarrow (\pi^{-1})_L \\ g^* h_* V & \xrightarrow{g^* \nu_h} & g^* h_! V \end{array}$$

Proof. We can split the square as

$$\begin{array}{ccccc} p_* q^* V & \xrightarrow{\gamma_p} & p_! q^* V & \xrightarrow{p_! \delta_p^{-1}} & p_! q^* V \\ \pi_R \uparrow & & \downarrow (\pi^{-1})_L & & \downarrow (\pi^{-1})_L \\ g^* h_* V & \xrightarrow{g^* \gamma_h} & g^* h_! V & \xrightarrow{g^* h_! \delta_h^{-1}} & g^* h_! V \end{array}$$

Since we are dealing with a homotopy pullback, which is exact by Proposition 2.11, the square on the left commutes by Proposition 3.8. We have to show that the multipliers δ on the right do not affect commutativity. This is a consequence of the above definition of Kan extensions: in fact at

each point both δ 's arise in the same way, from the following homotopy pullback

$$\begin{array}{ccccc} P' & \longrightarrow & P & \longrightarrow & N \\ \downarrow & & \downarrow & & \downarrow \\ \{*\} & \longrightarrow & M & \longrightarrow & Y \end{array}$$

□

Remark 4.10. Notice that, as a particular case, the above result states that Nakayama maps obtained from different choices of adjoints to a restriction functor are canonically conjugated.

In analogy with Theorem 3.9 we have

Theorem 4.11. *Let*

$$\begin{array}{ccc} & P & \\ p \swarrow & & \searrow q \\ M & \xRightarrow{\pi} & N \\ g \swarrow & & \searrow h \\ & Y & \end{array}$$

be a homotopy pullback of essentially finite groupoids and \mathcal{C} be a symmetric monoidal category with duals closed under essentially finite (co)limits, such that δ_p, δ_h are invertible. Let $\bar{\pi}: g_! p_! \rightarrow h_! q_!$ be the natural isomorphism induced by π . Then, the following diagram commutes

$$\begin{array}{ccccccc} g_! g^* & \xrightarrow{g_! \eta_*^p} & g_! p_* p^* g^* & \xrightarrow{g_! p_* \pi} & g_! p_* q^* h^* & \xrightarrow{g_! \nu_p} & g_! p_! q^* h^* \xrightarrow{\bar{\pi}} h_! q_! q^* h^* \xrightarrow{h_! \varepsilon_!^q} h_! h^* \\ & \searrow \eta_*^h & & & & & \nearrow h_! h^* \varepsilon_!^g \\ & & h_* h^* g_! g^* & \xrightarrow{\nu_h} & h_! h^* g_! g^* & & \end{array}$$

Proof. Being the square a homotopy pullback, we can take advantage of Proposition 2.11 and Proposition 4.9 above. The proof then proceeds exactly as in Theorem 3.9. □

Finally, as γ in Theorem 3.19, also ν can be seen to define a “monoidal transformation” between the “monoidal functors” given by left and right Kan extensions:

Theorem 4.12. *Let $f: M \rightarrow X$ and $g: N \rightarrow Y$ be maps, and \mathcal{C} be a symmetric monoidal category with duals closed under essentially finite (co)limits, such that δ_f, δ_g are invertible. Then for any $V \in \mathcal{C}^M, W \in \mathcal{C}^N$*

$$\begin{array}{ccc} (f \times g)_* (V \boxtimes W) & \xrightarrow{\mu_*} & f_* V \boxtimes g_* W \\ \nu_{f \times g} \downarrow & & \downarrow \nu_f \boxtimes \nu_g \\ (f \times g)_! (V \boxtimes W) & \xrightarrow{\mu_!} & f_! V \boxtimes g_! W \end{array}$$

commutes.

Proof. By Proposition 3.13, $\mu_!$ and μ_* are the left and right mates for diagrams of the form

$$\begin{array}{ccccc}
\mathcal{C}^X \times \mathcal{C}^Y & \xrightarrow{p_X^* \times p_Y^*} & \mathcal{C}^{X \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes} & \mathcal{C}^{X \times Y} \\
\downarrow f^* \times id & & \downarrow (f \times id)^* \times id & \nearrow \mu & \downarrow (f \times id)^* \\
\mathcal{C}^M \times \mathcal{C}^Y & \xrightarrow{p_M^* \times p_Y^*} & \mathcal{C}^{M \times Y} \times \mathcal{C}^{X \times Y} & \xrightarrow{\otimes (f \times id)^*} & \mathcal{C}^{M \times Y}
\end{array}$$

The left sided square commutes with ν by Proposition 4.9. For the right sided one, we can apply the same “splitting” trick as in Proposition 4.9 and, by Proposition 3.8, concentrate only on the factors δ ; since the only factor appearing is $\delta_{(f \times id)}$, which agrees with tensor product, commutativity is verified. \square

4.2. Nakayama Categories. We can finally collect the results obtained so far, and characterise the kind of categories we should feed our quantization with. It is clear that the main issue was functoriality of the map γ . In view of the linear case, we have introduced the weights δ and the new map ν . This in fact can be seen (Theorem 6.3) to solve our problem over *Vect*, in addition to recovering and formalizing results and definitions from the previous literature [BD2, FHLT, Mo1, Mo2]. We will therefore ask our target category \mathcal{C} to guarantee existence and functoriality of the Nakayama map as follows:

Definition 4.13. *We say that a category \mathcal{C} is a Nakayama Category if*

- i) \mathcal{C} has essentially finite colimits.
- ii) \mathcal{C} is symmetric monoidal.
- iii) \mathcal{C} has duals.
- iv) For any map $X \xrightarrow{f} Y$ in \mathbf{Gpd}_{fin} , the endomorphism δ_f is invertible.
- v) For any composable pair of maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathbf{Gpd}_{fin} the diagram

$$\begin{array}{ccc}
& g_! f_* & \\
\nu_g \nearrow & & \searrow g_! \nu_f \\
g_* f_* & \xrightarrow{\nu_{gf}} & g_! f_! \\
g_* \nu_f \searrow & & \nearrow \nu_g \\
& g_* f_! &
\end{array}$$

commutes.

Let us further motivate the above definition. Essentially, we would like to apply the machinery developed in the previous sections to restriction functors $f^*: \mathcal{C}^Y \rightarrow \mathcal{C}^X$ and their adjoints, given by Kan extensions. Since we are dealing with groupoids points ii) and iii) ensure, by Proposition 2.2, that the categories \mathcal{C}^X we will consider are symmetric monoidal with duals. The restriction to essentially finite groupoids in point i) is motivated by concrete examples of categories with duals, such as vector spaces, where infinite colimits are not dualizable objects. Point i) ensures the existence of left Kan extensions (along maps of essentially finite groupoids) and therefore of adjoint pairs of functors $f_! \dashv f^*$ between \mathcal{C}^M and \mathcal{C}^X , for any $M \xrightarrow{f} X$. Being

the restriction functors (symmetric) monoidal we can now apply Proposition 4.1, and conclude that also right Kan extensions exist. Points $i) - v)$, altogether, will allow us to use to results of the previous pages so to define the quantization functors on objects and, via the Nakayama map, on morphisms. In particular, point $v)$ will be necessary in Theorems 5.1 and 5.2, to verify functoriality of *Sum* and *Prod* by taking advantage of Theorem 4.11.

Remark 4.14. In view of the above comments and Proposition 4.1, one can reformulate Definition 4.13, by asking for (essentially finite) limits instead of colimits. The two formulations are of course equivalent.

Remark 4.15. Notice that condition $v)$ of Definition 4.13, and in general the definition of ν_f , does not depend on the choice of representatives for the Kan extensions along f . In fact, by Remark 4.10, Nakayama maps relative to isomorphic pairs of functors are conjugated.

5. QUANTIZATION

The techniques developed in the previous sections will now be used to define, in terms of Kan extensions, two monoidal functors

$$Sum: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

$$Prod: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

from a category of representations of essentially finite groupoids valued in any Nakayama category \mathcal{C} .

We'll see in section 5.3 that the two functors are canonically isomorphic, thus implying invertibility of the Nakayama morphism ν .

5.1. A monoidal category of local systems. Here below we will briefly describe the source category $\text{Fam}_1^1(\mathcal{C})$ of the quantization functors.

In the general case, this is an (∞, n) -category $\text{Fam}_n(\mathcal{C})$ “of spans of ∞ -groupoids”. A typical object is an ∞ -groupoid X together with a functor $X \rightarrow \mathcal{C}$ to the target (∞, n) -category \mathcal{C} . The 1-morphisms are given by spans $X \leftarrow M \rightarrow Y$ along with a filling cell making the resulting diagram commute. A 2-morphism from M to M' is given by a span of ∞ -groupoids $M \leftarrow N \rightarrow M'$, with a filling two-cell; k -morphisms until level n are similarly given by spans between spans, while higher ($k > n$) k -morphisms are given by (higher) homotopies between maps of ∞ -groupoids, compatible with the underlying diagrams. A precise description of such (∞, n) -category $\text{Fam}_n(\mathcal{C})$ has been given in [Ha2]; there the author uses the notation $\text{Span}_n(\mathcal{S}; \mathcal{C})$, while we preferred to keep the one used in [FHLT] and [Lu1].

In our situation, being \mathcal{C} merely a category, we need to restrict our attention to (finite) homotopy 1-types (i.e. essentially finite groupoids) in order to use the results of section 4. Also, we can avoid working with higher categories and just consider the homotopy category of $\text{Fam}_1^1(\mathcal{C})$ as a source. By an abuse of notation, this will again be denoted by $\text{Fam}_1^1(\mathcal{C})$, and we will construct functors

$$\text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

One can anyway check that the two functors lift to ∞ -functors (where now the target will be the classifying diagram [Re] of \mathcal{C}).

An object of $\text{Fam}_1^1(\mathcal{C})$ will be given, as already mentioned, by an essentially finite groupoid X , and a functor

$$X \xrightarrow{V} \mathcal{C}$$

Morphisms will be (equivalence classes of) diagrams

$$\begin{array}{ccc} & M & \\ f \swarrow & & \searrow g \\ X & \xRightarrow{\alpha} & Y \\ v \searrow & & \swarrow w \\ & \mathcal{C} & \end{array}$$

with composition given by (an equivalence class of) homotopy pullback along the middle maps

$$\begin{array}{ccccc} & & P & & \\ & p \swarrow & & \searrow q & \\ & M & \xRightarrow{\pi} & N & \\ f \swarrow & & & & \searrow k \\ X & \xRightarrow{\alpha} & Y & \xRightarrow{\beta} & Z \\ & U \searrow & & \swarrow W & \\ & & \mathcal{C} & & \end{array}$$

and the identity morphism is clearly given by

$$\begin{array}{ccc} & X & \\ id \swarrow & & \searrow id \\ X & \xRightarrow{id} & X \\ v \searrow & & \swarrow v \\ & \mathcal{C} & \end{array}$$

Cartesian product and external tensor product (Definition 3.12) endow $\text{Fam}_1^1(\mathcal{C})$ with the structure of a symmetric monoidal category with duals.

Given objects $X \xrightarrow{V} \mathcal{C}, Y \xrightarrow{W} \mathcal{C}$, their tensor product is

$$X \times Y \xrightarrow{V \boxtimes W} \mathcal{C}$$

The unit is provided by the unit of \mathcal{C} , viewed as a functor

$$* \xrightarrow{1} \mathcal{C}$$

from the one-point space.

Finally, the dual to $X \xrightarrow{V} \mathcal{C}$ is given by $X \xrightarrow{V^d} \mathcal{C}$ as from Proposition 2.2, with coevaluation and evaluation morphism

$$\begin{array}{ccc} X & & X \\ \swarrow \Delta & & \swarrow \Delta \\ * & \xRightarrow{co} & X \times X \\ \searrow 1 & & \searrow V \boxtimes V^d \\ & \mathcal{C} & \end{array} \quad \begin{array}{ccc} X & & X \\ \swarrow \Delta & & \swarrow \Delta \\ X \times X & \xRightarrow{ev} & * \\ \searrow V^d \boxtimes V & & \searrow 1 \\ & \mathcal{C} & \end{array}$$

5.2. The quantization functors. It is not difficult, at this point, to define the functors Sum and $Prod$. As already mentioned in the introduction the value on objects will be given by (co)limits, while the map ν will be used to define the functors on morphisms. The results of section 3.3 and the compatibility with ν proved in Theorem 4.12 will make them into symmetric monoidal functors.

In the following we will work up to the obvious equivalence $\mathcal{C} \simeq \mathcal{C}^*$. For a space X , denote by x the unique map $X \rightarrow *$ to the point.

Theorem 5.1. *Let \mathcal{C} be a Nakayama category. Then, there exists a symmetric monoidal functor*

$$Sum: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

Proof. Given an object $V: X \rightarrow \mathcal{C}$ of $\text{Fam}_1^1(\mathcal{C})$, we let

$$Sum V = x_! V$$

be the left Kan extension of V along x , i.e. its colimit.

To a morphism

$$\begin{array}{ccc} & M & \\ f \swarrow & & \searrow g \\ X & \xRightarrow{\alpha} & Y \\ v \swarrow & & \searrow w \\ & \mathcal{C} & \end{array}$$

Sum associates the map $Sum(\alpha)$

$$x_! V \xrightarrow{x_! \eta_*^f} x_! f_* f^* V \xrightarrow{x_! f_* \alpha} x_! f_* g^* W \xrightarrow{x_! \nu_f} x_! f_! g^* W \xrightarrow{\cong} y_! g_! g^* W \xrightarrow{y_! \varepsilon_1^g} y_! W$$

It is clear that on identity morphisms, i.e. squares

$$\begin{array}{ccc} & X & \\ id \swarrow & & \searrow id \\ X & \xRightarrow{\alpha} & X \\ v \swarrow & & \searrow v \\ & \mathcal{C} & \end{array}$$

the above assignment reduces to the identity map.

Now let's consider a composition of morphisms in $\text{Fam}_1^1(\mathcal{C})$, that is a diagram

$$\begin{array}{ccccc} & & P & & \\ & p \swarrow & & \searrow q & \\ & M & \xRightarrow{\pi} & N & \\ f \swarrow & & & & \searrow k \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Z \\ \alpha \swarrow & & \searrow \beta & & \\ & \mathcal{C} & & & \end{array}$$

$U \swarrow \quad \downarrow V \quad \searrow W$

where the upper square is (by definition of $\text{Fam}_1^1(\mathcal{C})$) a homotopy pullback of 1-types. The outer square and the two diagrams labeled by α and β give rise to 1-morphisms $Sum(\beta\pi\alpha)$, $Sum(\alpha)$, $Sum(\beta)$ of \mathcal{C} . I claim that the first is the composition of the other two.

Noting that the isomorphism $x_! f_! p_! \rightarrow z_! k_! q_!$ is the composition

$$x_! f_! p_! \xrightarrow{\cong} y_! g_! p_! \xrightarrow{y_! \tilde{\pi}} y_! h_! q_! \xrightarrow{\cong} z_! k_! q_!$$

a simple analysis of the resulting diagram using the functoriality hypothesis and Theorem 4.11 proves the claim.

It remains to show that Sum is actually a symmetric monoidal functor. The isomorphism $\mu_!$ of Proposition 3.13 provides an isomorphism

$$\mu_!: Sum(V \boxtimes V') \rightarrow Sum(V) \otimes Sum(V')$$

Naturality of $\mu_!$, i.e. commutativity of

$$\begin{array}{ccc} Sum(V \boxtimes V') & \xrightarrow{\mu_!} & Sum(V) \otimes Sum(V') \\ \downarrow Sum(\alpha \boxtimes \alpha) & & \downarrow Sum(\alpha) \otimes Sum(\alpha') \\ Sum(W \boxtimes W') & \xrightarrow{\mu_!} & Sum(W) \otimes Sum(W') \end{array}$$

for morphisms α, α' and the corresponding tensor product

$$\begin{array}{ccccc} \begin{array}{ccc} & M & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{\alpha} & Y \\ v \searrow & & \swarrow w \\ & \mathcal{C} & \end{array} & \begin{array}{ccc} & M' & \\ f' \swarrow & & \searrow g' \\ X' & \xrightarrow{\alpha'} & Y' \\ v' \searrow & & \swarrow w' \\ & \mathcal{C} & \end{array} & \begin{array}{ccc} & M \times M' & \\ f \times f' \swarrow & & \searrow g \times g' \\ X \times X' & \xrightarrow{\alpha \boxtimes \alpha'} & Y \times Y' \\ v \boxtimes v' \searrow & & \swarrow w \boxtimes w' \\ & \mathcal{C} & \end{array} \end{array}$$

follows from Propositions 3.16 to 3.18 and Theorem 4.12.

The unit isomorphism

$$Sum(1) \rightarrow 1$$

is just the identity, being $Sum(1)$ simply the Kan extension along the identity map $* \rightarrow *$.

Finally associativity, unitality and commutativity of Sum follow by Propositions 3.14 and 3.15. \square

As the reader might have noticed, the arguments used in the previous theorem can all be dualized. It is natural then to consider right Kan extensions in place of left ones, and try to define a “limit” version of Sum . To further stress this duality, recall from Remark 4.14 that we can define a Nakayama category as having (essentially finite) limits instead of colimits.

A similar proof gives

Theorem 5.2. *Let \mathcal{C} be a Nakayama category. Then, there exists a symmetric monoidal functor*

$$Prod: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

Proof. The proof is dual to the one of Theorem 5.1. This time, $Prod$ associates to $V: X \rightarrow \mathcal{C}$ the right Kan extension $x_* V$ (i.e. the limit of V).

A morphism

$$\begin{array}{ccc} & M & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{\alpha} & Y \\ v \searrow & & \swarrow w \\ & \mathcal{C} & \end{array}$$

is taken to the map

$$x_*V \xrightarrow{x_*\eta_*^f} x_*f_*f^*V \xrightarrow{\cong} y_*g_*f^*V \xrightarrow{y_*g_*\alpha} y_*g_*g^*W \xrightarrow{y_*\nu_g} y_*g!g^*W \xrightarrow{y_*\varepsilon_!^g} y!W$$

The multiplication isomorphism

$$Prod(V \boxtimes V') \rightarrow Prod(V) \otimes Prod(V')$$

is clearly given by the isomorphism μ_* of Proposition 3.13. \square

5.3. Siamese twins. We can finally conclude by comparing the two functors just defined. Recalling from Theorem 4.12 the “monoidal” behaviour of the transformation ν , the following comes with no surprise

Proposition 5.3. *Let \mathcal{C} be a Nakayama category. Then the Nakayama map defines a monoidal transformation*

$$\nu: Prod \rightarrow Sum$$

Proof. First, we check that the maps

$$\nu_x: Prod(V) = x_*V \rightarrow x!V = Sum(V)$$

define a natural transformation. For any span

$$\begin{array}{ccc} & M & \\ f \swarrow & & \searrow g \\ X & \xrightarrow{\alpha} & Y \\ V \searrow & & \swarrow W \\ & \mathcal{C} & \end{array}$$

we obtain in fact by the functoriality assumption and Proposition 4.9, a commutative diagram

$$\begin{array}{ccccccccccc} x_*V & \xrightarrow{x_*\eta_*^f} & x_*f_*f^*V & \xrightarrow{x_*f_*\alpha} & x_*f_*g^*W & \xrightarrow{id_R} & y_*g_*g^*W & \xrightarrow{y_*\nu_g} & y_*g!g^*W & \xrightarrow{y_*\varepsilon_!^g} & y_*W \\ \downarrow \nu_x & & & & \downarrow \nu_x & \searrow \nu_{xf} & \downarrow \nu_{yg} & & \downarrow \nu_y & & \downarrow \nu_y \\ x!V & \xrightarrow{x!\eta_*^f} & x!f_*f^*V & \xrightarrow{x!f_*\alpha} & x!f_*g^*W & \xrightarrow{x!\nu_f} & x!f!g^*W & \xrightarrow{id_L} & y!g!g^*W & \xrightarrow{y!\varepsilon_!^g} & y!W \end{array}$$

exhibiting ν as a natural transformation.

The fact that ν is monoidal, follows by simply applying Theorem 4.12 to

$$\begin{array}{ccc} (x \times x')_*(V \boxtimes V') & \xrightarrow{\nu} & (x \times x')_!(V \boxtimes V') \\ \downarrow \mu_* & & \downarrow \mu_! \\ x_*V \otimes x'_*V' & \xrightarrow{\nu \otimes \nu} & x!V \otimes x'!V' \end{array}$$

and trivially from

$$\begin{array}{ccc} & 1 & \\ id \swarrow & & \searrow id \\ id_*1 & \xrightarrow{\nu} & id!1 \end{array}$$

So that ν is compatible with the monoidal structure of $Prod$ and Sum . \square

We can summarize the previous section and the above result in a single theorem:

Theorem 5.4. *Let \mathcal{C} be a Nakayama category, then there exist canonically isomorphic symmetric monoidal functors*

$$Prod: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

$$Sum: \text{Fam}_1^1(\mathcal{C}) \rightarrow \mathcal{C}$$

respectively defined by right and left Kan extensions of local systems.

Proof. We already know that the Nakayama map defines a monoidal transformation $Prod \rightarrow Sum$. Being $\text{Fam}_1^1(\mathcal{C})$ a category with duals, it follows from Corollary 2.5 that ν is actually an isomorphism. \square

As a consequence, we obtain that in any Nakayama category \mathcal{C} limits and colimits are canonically isomorphic, via either ν or γ , so that the assumption in [FHLT, Mo1, Mo2] is now a corollary. Of course, this extends to an isomorphism between left and right Kan extensions, generalising a well-known fact from representation theory.

Corollary 5.5. *Let \mathcal{C} be a Nakayama category. Then, for any map $X \xrightarrow{f} Y$ of essentially finite groupoids, the Nakayama map defines a canonical isomorphism*

$$f_*V \rightarrow f_!V$$

for any $V: X \rightarrow \mathcal{C}$.

Proof. We know that the statement is true when f is the terminal map to the point, so that the right and left Kan extensions are actually the limit and colimit functors. From Definition 4.6 it follows that for any map $X \xrightarrow{f} Y$, the value of ν_f at any point $y \in Y$ is an isomorphism as well. \square

6. LINEAR REPRESENTATIONS OF GROUPOIDS

We conclude by rapidly considering the results of the previous sections in the context of finite vector spaces. While in this situation the functors Sum and $Prod$ themselves do not give rise to really interesting theories, the techniques used to build them and in particular Corollary 5.5 specialize to relevant known results, which appear now under a different light.

Recall that a representation V of a group G is the same thing as a functor $V: BG \rightarrow Vect$ from the classifying space BG of G . Similarly, a $Vect$ -valued representation of a groupoid X is a functor $V: X \rightarrow Vect$.

Also notice that any essentially finite groupoid X is equivalent to a finite disjoint union $\coprod BG_i$, with the G_i 's finite groups. Therefore we will simplify notations (and calculations) by assuming that $Map(x, y)$ is empty if $x \neq y$, and denote $A_x = Map(x, x)$.

In the following $Vect$ will mean finite (hence dualizable) vector spaces over a field k of characteristic zero.

The usual presentation of induction and coinduction of group representations in terms of tensor and hom spaces then naturally generalises to essentially finite groupoids:

Theorem 6.1. *Let $f: X \rightarrow Y$ be a morphism between essentially finite groupoids, and let $V: X \rightarrow \text{Vect}$ be a representation of X . Then*

- *The value of $f_!V$ at $y \in Y$ is*

$$f_!V(y) = \bigoplus_{x|fx=y} k[A_y] \otimes_{k[A_x]} V(x)$$

- *At $x \in X$, the unit $\eta: V(x) \rightarrow f^*f_!V(x) = f_!V(fx)$ is the map*

$$v \mapsto 1 \otimes_{k[A_x]} v \in k[A_y] \otimes_{k[A_x]} V(x)$$

- *The counit $\varepsilon: f_!f^*W(y) \rightarrow W(y)$ is the map induced by the assignments*

$$k[A_y] \otimes_{k[A_x]} W(y) \ni g \otimes_{k[A_x]} w \mapsto gw$$

where $g \in A_y$

Theorem 6.2. *Let $f: X \rightarrow Y$ be a morphism between essentially finite groupoids, and let $V: X \rightarrow \text{Vect}$ be a representation of X . Then*

- *The value of f_*V at $y \in Y$ is*

$$f_*V(y) = \bigoplus_{x|fx=y} \text{Hom}_{k[A_x]}(k[A_y], V(x))$$

- *At $y \in Y$, the unit $\eta: W(y) \rightarrow f_*f^*W(y)$ is the map*

$$w \mapsto \sum_{x|fx=y} \phi_{w,x}$$

where $\phi_{w,x}(g) = gw$ for $g \in A_y$.

- *The counit $\varepsilon: f^*f_*V(x) = f_*V(fx) \rightarrow V(x)$ is the map*

$$\sum_{x'|fx'=fx} \phi_{x'} \mapsto \phi_x(1)$$

One can now compute the Nakayama morphism for a map f of essentially finite groupoids, according to the above presentations of Kan extensions.

Theorem 6.3. *Let $f: X \rightarrow Y$ be a morphism between essentially finite groupoids, and $V: X \rightarrow \text{Vect}$ a representation of X . For $x \in X$ let $f(A_x)$ be the image of A_x in A_{fx} , and denote by K_x the kernel of the induced map $A_x \xrightarrow{f} A_{fx}$. Then*

- *The map γ_f at $y \in Y$*

$$\gamma_{f,y}: \bigoplus_{x|fx=y} \text{Hom}_{k[A_x]}(k[A_y], V(x)) \rightarrow \bigoplus_{x|fx=y} k[A_y] \otimes_{k[A_x]} V(x)$$

is induced by the assignments

$$\text{Hom}_{k[A_x]}(k[A_y], V(x)) \ni \phi_x \mapsto \frac{1}{|f(A_x)|} \sum_{g \in A_y} g^{-1} \otimes \phi_x(g)$$

- *The map $\delta_f: f_!V(y) \rightarrow f_!V(y)$ at y is given at each summand by multiplication by*

$$|K_x|$$

- The Nakayama map at y is induced at each summand by the map

$$\phi_x \mapsto \frac{1}{|A_x|} \sum_{g \in A_y} g^{-1} \otimes \phi_x(g)$$

As we already anticipated, we have that

Theorem 6.4. *The category of finite dimensional vector spaces over a zero-characteristic field is a Nakayama category.*

Proof. We know that the category of finite dimensional vector spaces is symmetric monoidal with duals, and it has essentially finite colimits; if the characteristic of the field is zero, then δ_f is invertible for any f . The functoriality of the Nakayama map follows from a tedious but not difficult computation. \square

With Theorems 6.3 and 6.4 at hand, one can easily check that in the case of trivial representations of groupoids the above recovers the values in [Mo1]. In particular, invertibility of the Nakayama morphism and ambidexterity of the adjunction given by restriction appear now as a consequence rather than an ingredient to quantization. Corollary 5.5 in fact specializes to

Theorem 6.5. *Let k be a zero-characteristic field, $H \xrightarrow{f} G$ a morphism of finite groups, and $\text{coInd}_H^G, \text{Ind}_H^G : \text{Rep}_k(H) \rightarrow \text{Rep}_k(G)$ the coinduction and induction functors. Then*

- i) *There is a canonical natural isomorphism $\nu : \text{coInd}_H^G \rightarrow \text{Ind}_H^G$, the Nakayama isomorphism*
- ii) *Induction is both a left and a right adjoint to restriction*

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